Making Money: Existence and Determination of Commodity Money in General Equilibrium

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April 2021

Abstract

The classic Arrow-Debreu (1954) general equilibrium model cannot sustain or account for the existence of money. This lacuna arises because each household and firm faces a single budget constraint summarizing revenue and expense in all commodities. Money, a carrier of value between transactions, has no function when all credits and debits are rolled into a single expression.

A trading post model of $N \geq 3$ commodities and transaction costs generates $\frac{1}{2}N(N-1)$ separate budget constraints with distinct bid and ask prices. General equilibrium, market-clearing prices and transactions at each trading post, exists under conventional continuity and convexity conditions. Commodities acquired by an agent at one trading post and disbursed at another constitute commodity money.

Keywords: Arrow-Debreu, barter, bid ask spread, commodity money, convex, general equilibrium, Kakutani Fixed Point Theorem, upper hemicontinuity, lower hemicontinuity, thick market externality, trading post, transaction cost

*An early version of this paper was presented as the Rolf Mantel lecture to the 2017 meeting of the Argentine Association of Political Economy at Bariloche Argentina. The support of University of California San Diego Academic Senate research grants has been most helpful. It is a pleasure to acknowledge the hospitality of the Research Department of the Federal Reserve Bank of San Francisco and of the Stanford University Economics Department. Professors Brendan Beare, William Brainard, Gabriele Camera, Marjorie Flavin, Duncan Foley, Jerry Green, Dror Goldberg, James Hamilton, Ron Harstad, Ronald McKinnon, Augusto Nieto-Barthaburu, Valerie Ramey, Giacomo Rondina, Joel Sobel, Alexis Toda, Johannes Wieland, Ms. Adele Barsh and Ms.W. Shaffer have all made helpful comments, but bear no responsibility for remaining errors.

1 Where's the Money?

Describing a research agenda, Wright (2018) comments " the use of money ... in facilitating exchange ... should be outcomes of ... our theories." See also (Hellwig 1993).

Hahn (1982) writes "The...challenge that...money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu."

The budget constraint in most conventional microeconomics does not lead to monetary trade. A household has an endowment $r \in R_+^N$ and a consumption plan $c \in R_+^N$ formed at prevailing prices $p \in R_+^N$. Then the choice of c is constrained by the budget, $p \cdot c \leq p \cdot r$. This is precisely the formulation in the Arrow-Debreu model, Arrow & Debreu (1954),Debreu (1959). We know from everyday experience that there is not one single budget constraint but many. At each of many transactions, an agent pays for what he gets. To introduce money into the Arrow-Debreu model, the model needs to introduce a multiplicity of separate budget constraints.

A century earlier, Menger (1892) explained:

> A commodity should be given up by its owner in exchange for another more useful to him. But ... exchange ... for little metal disks apparently useless as such, or for documents representing the latter, is ... mysterious.
>
> Goods [are] ...more or less saleable [*absatzfahig* marketable], according to the greater or less facility with which they can be disposed of ... at current purchasing prices, or with less or more diminution.
>
> The theory of money necessarily presupposes a theory of the saleableness [*Absatzfahigkeit*, marketability] of goods.
>
> when any one has brought goods not highly saleable to market, the idea uppermost in his mind is to exchange them, not only for such as he happens to be in need of, but ... for other goods ... more saleable than his own .... By ... a mediate exchange, he gains the prospect of accomplishing his purpose more surely and economically than if he had confined himself to direct exchange .... Men have been led ... without convention, without legal compulsion,... to exchange ... their wares ... for other goods ... more saleable ... which ... have ... become generally acceptable media of exchange.
>
> legislation...is neither the only, nor the primary mode in which money has taken its origin...Money has not been generated by law. In its origin it is a social, and not a state institution.

The plan of this article is to demonstrate an economic general equilibrium where (commodity) money arises endogenously. Goods trade as commodity pairs where $N \geq 3$ commodities generate $\frac{1}{2}N(N-1)$ separate budget constraints. There are distinct bid and ask prices. 'Saleability' is a narrow bid/ask spread. Market-clearing prices and transactions are shown to exist based on continuity and convexity conditions. Commodities acquired by a household at one trading post and disbursed at another are commodity money. As Menger asserted, use of media of exchange is an outcome of market equilibrium.

2 Trading Posts

The trading post model consists of N commodities traded pairwise at $\frac{1}{2}N(N-1)$ trading posts, each with a *quid pro quo* budget constraint. Acquistions are repaid by delivery of equal value. There are distinct bid and ask prices; the bid/ask spread reflects transaction costs. Walras (1896) forms the picture this way:

we shall imagine that the place which serves as a market for the exchange of all the commodities... for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have $\frac{m(m-1)}{2}$ special markets each identified by a signboard indicating the names of the two commodities exchanged there as well as their ... rates of exchange...

The determination of which trading posts are active in equilibrium is endogenous and characterizes the monetary or barter character of trade. When most of the $\frac{1}{2}N(N-1)$ trading posts are active, the equilibrium is barter. When a household acquires a commodity at one trading post and disposes of it at another, the commodity is acting as commodity money. The equilibrium is monetary with a unique money if only N trading posts out of $\frac{1}{2}N(N-1)$ are active, those trading all goods against 'money,' (Starr 2003, 2012).

3 Commodities, Prices, Bid/Ask Spread

Let $N =$ number of elementary commodities, $N \geq 3$, each of which may trade against any other, generating

$N(N-1)$ dimensions of activity at bid prices and another $N(N-1)$ at ask prices. There are $\frac{1}{2}N(N-1)$ trading posts with two goods traded at each. Further, commodities enter both in bid and ask price transactions (wholesale or retail transactions). In addition, commodities act as inputs to transaction costs.

Price vectors are q, π each $\in \mathbb{R}_+^{N(N-1)}$. Eventually, homogeneity of degree zero of supply and demand correspondences will be demonstrated, so that attention can be confined to the unit simplex in $\mathbb{R}_+^{2N(N-1)}$. q represents the prevailing bid prices, $q+\pi$ is the vector of prevailing ask prices, π is the vector of the premium above bid prices of ask prices.

A typical co-ordinate entry will be denoted $x(k,l), 1 \leq k,l \leq N, k \neq l$, representing commodity k at the trading post of k for l. This is the same trading post as for l for k. $x^S(k,l)$ represents good k traded for l at bid price. $x^B(k,l)$ represents good k traded for l at ask price. The bid price of (k,l) is $q(k,l)$; the ask price is $q(k,l) + \pi(k,l)$. $\pi(k,l)$ is the ask premium or retail premium. Purchases are positive co-ordinates, sales are negative. Much of this notation follows Foley (1970).

Let:

$$B(p) \equiv \{x \in R^{2N(N-1)} | \; \forall k, l \in \{1 \leq k, l \leq N, k \neq l\}$$
$$q(k,l)x^S(k,l) + (q(k,l) + \pi(k,l))x^B(k,l)$$
$$+ q(l,k)x^S(l,k) + (q(l,k) + \pi(l,k))x^B(l,k) \leq 0\}$$

$B(p)$ is the trading budget presenting the collection of transactions priced for *quid pro quo* at each trading post. Budgets are balanced at each trading post separately. Purchases enter with positive signs, sales with negative signs. Then the budget constraint is that sales must be (weakly) more valuable than purchases. The budget constraint in this model distinguishes it from the Arrow-Debreu model. Each agent in this setting faces $\frac{1}{2}N(N-1)$ budget constraints, one at each trading post.

H is the finite population of households. For each $i \in H$, let $x^i \in R^{2N(N-1)}$ be i's trade vector. Household i has an endowment $r^i \in R^{2N(N-1)}$. The dimension of r^i is set at $2N(N-1)$ for notational consistency. Household i has utility function u^i . Households sell at bid prices, q, and buy at ask prices, $q + \pi$.

4 Example

Consider a simple example where there are only 3 types of households and 3 commodities. Each type of household is identified by its preferences and there are a large number, Q of each type. As such, each household will act as a price taker and will not be able to meaningfully influence transaction costs. For a household, a trading plan is a vector $x \in \mathbb{R}^{12}$. Households are labeled with superscript $h \in \{i, j, k\}$. Household h is endowed with 100 units of commodity h. Households' utility functions are as follows:

$$\mathcal{U}^i(x) = x^{iB}(k,i) + x^{iB}(k,j) + x^{iS}(k,i) + x^{iS}(k,j)$$
$$\mathcal{U}^j(x) = x^{jB}(i,j) + x^{jB}(i,k) + x^{jS}(i,j) + x^{jS}(i,k)$$
$$\mathcal{U}^k(x) = x^{kB}(j,i) + x^{kB}(j,k) + x^{kS}(j,i) + x^{kS}(j,k)$$

Household i is endowed with i and wants k, j is endowed with j and wants i, k is endowed with k and wants j. There are three trading posts, $\{i,j\}, \{j,k\}, \{k,i\}$. At each trading post, each commodity there has a bid price of unity. $q(i,j) = q(j,i) = q(j,k) = q(k,j) = q(k,i) = q(i,k) = 1$. There is a transaction cost markup at each post, particularly high at $\{i,j\}$. $\pi(i,j) = \pi(j,i) = 1$, $\pi(i,k) = \pi(k,i) = \pi(k,j) = \pi(j,k) = \epsilon$. $\epsilon > 0$, ϵ is small.

Ask prices at the trading posts then are $q(i,j)+\pi(i,j) = 1+1 = 2 = q(j,i)+\pi(j,i)$, $q(i,k) + \pi(i,k) = 1 + \epsilon = q(k,i) + \pi(k,i)$, $q(j,k) + \pi(j,k) = 1 + \epsilon = q(k,j) + \pi(k,j)$, where $\epsilon > 0$, ϵ is small.

Households of type j could trade directly for i, but they would incur the 100% markup on i. Trading indirectly through k doubles the trading volume but incurs

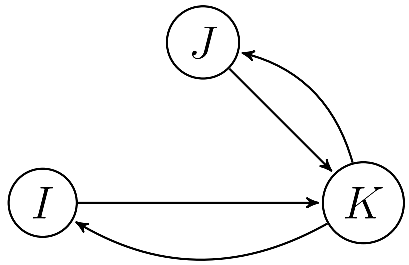

Figure 1: Trade Diagram for 3 Households

only two ϵ markups. So trading through k is advantageous. k becomes commodity money.

Figure 1 offers a pictorial representation of the equilibrium behavior.

Hence, for such a pattern to arise, it must be that type j's utility from indirect trade is greater than its utility from direct trade and the opposite should hold for types i and k.

5 Firms

There is a finite population of firms $j \in F$. Firms deal in the trading process, buying and selling, incurring transaction costs in commodities. There is no production in the model. Firm j formulates a transaction plan $(y^{jS}, y^{jB}, w^j) \in R^{3N(N-1)}$. Positive co-ordinates of y^{jB}, y^{jS} indicate purchases. Negative co-ordinates indicate sales. Negative co-ordinates in w indicate inputs to the trading technology, transaction costs. y^{jS} is the vector of transactions, purchases and sales, the firm makes at bid (wholesale) prices. y^{jB} is the vector of purchases and sales subject to the premium buying (retail) price. Note that in contrast to the households, for the firm, both y^{jS} and y^{jB} may have both positive and negative co-ordinates. The firm's ability or inability to deal in positive or negative actions at bid or ask prices is formalized in its technology Y^j. The budget constraint on firm transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$,

$$
\begin{aligned}
&q(k,\ell) \cdot y^{jS}(k,\ell) + (q(k,\ell) + \pi(k,\ell)) \cdot y^{jB}(k,\ell) \\
&+ (q(\ell,k) + \pi(\ell,k)) \cdot y^{jB}(\ell,k) + q(\ell,k) \cdot y^{jS}(\ell,k) \le 0 \quad (\mathcal{B}')
\end{aligned}
$$

Equivalently:

$$
\begin{aligned}
(y^{jS}, y^{jB}) \in B(p) \equiv \{x \in R^{2N(N-1)} | & q(k,l)x^S(k,l) + (q(k,l) + \pi(k,l))x^B(k,l) \\
& + q(l,k)x^S(l,k) + (q(l,k) + \pi(l,k))x^B(l,k) \le 0, \\
& \text{for all } 1 \le k, l \le N, k \ne l\}
\end{aligned}
$$

A suitable maximand for the firm needs to be defined. It is simplest to ask the firm to maximize profit, $(q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j)$. This applies though the firm cannot easily distribute the accounting value of profit to shareholders.

The technically possible mix (y^{jS}, y^{jB}, w^j) of purchases, inputs, and sales of firm j is contained in the closed convex set $Y^j \subseteq R^{3N(N-1)}$. Firm j's supply decision then is:

$$S^{j\dagger}(p) \equiv \{(y^S, y^B, w)|(y^S, y^B, w) = \arg\max\ (q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j)$$
$$\text{subject to } (y^S, y^B, w) \in (Y^j \cap [B(p) \times \mathbb{R}^{N(N-1)}])\}$$

Firm j's marketed supply behavior then is:

$$S^j(p) \equiv \{(y^S, y^B)|(y^S, y^B, w) = \arg\max\ (q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j)$$
$$\text{subject to } (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}])\}$$

There is a finite set of households, H. For each $i \in H$, let $x^i \in \mathbb{R}^{2N(N-1)}$ represent i's transaction offers. Define $\hat{x} = (x^1, x^2, ..., x^{\#H}) \in R^{\#H2N(N-1)}$. Household i owns a proportion Θ^{ij} of firm j (Foley notation), $1 \geq \Theta^{ij} \geq 0$, $\sum_{i \in H} \Theta^{ij} = 1$. The distribution of firms' earnings to households cannot easily be summarized as a money dividend, so it will enter as a commodity dividend distribution from firms to households, $\Theta^{ij}(y^{jS}, y^{jB} + w^j)$.

The following assumptions on the trading technology, P.I to P.IV, are standard in the general equilibrium literature: they are adapted from Starr (2011)

(P.I) Y^j is convex for all j.
(P.II) $0 \in Y^j$ for each j.
(P.III) Y^j is a closed convex cone for all j.
(P.IV) (No Free Transaction) Let $(y^S, y^B, w) \in Y^j$ then:
(i)If $(y^S, y^B, w) \neq 0$ then $w \neq 0, w \leq 0$ co-ordinatewise.
(ii) for each $k = 1, 2, \ldots, N$, $\sum_{\ell} y^S(k, \ell) + \sum_{\ell} y^B(k, \ell) + \sum_{\ell} w(k, \ell) \geq 0$.

Note that P.IV implies that there are no free transactions. It also implies limits on reversibility; some firms' actions in the trading sector may reverse those of another firm, but transaction costs will be irreversibly expended in the process. P.IV(ii) says that firm j must arrange its affairs so that it is (weakly) a net purchaser of each of the N commodities. Deliveries (negative) at one trading post may exceed purchases (positive) there, but aggregate purchases by the firm of any commodity must (weakly) exceed sales. This reflects that the economy is pure exchange with some resource expenditure on transaction costs.

6 Attainable Transactions

The aggregate trading technology is $Y \equiv \sum_{j \in F} Y^j$. The economy's initial resource vector is

$r = \sum_{i \in H} r^i \in R_+^{2N(N-1)}$. Then $(y^S, y^B, w) \in Y$ is said to be attainable if for each $k = 1, 2, \ldots, N$, we have:

$$\sum_{\ell} [y^S(k,\ell) + y^B(k,\ell)] \leq \sum_{\ell} r(k,\ell), \text{and}$$
$$\sum_{\ell} [w(k,\ell)] \geq -\sum_{\ell} r(k,\ell)$$

$(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ is said to be attainable in $Y^{j\prime}$ if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ for all $j \in F, j \neq j\prime$, so that $(y^{j\prime S}, y^{j\prime B}, w^{j\prime B}) + \sum_{j \in F, j \neq j\prime} (y^{jS}, y^{jB}, w^j)$ is attainable.

$(y^{jS}, y^{jB}) \in S^j(p)$ is said to be attainable if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ so that (y^{jS}, y^{jB}, w^j) is attainable in Y^j.

Lemma 1. *Assume P.I through P.IV. Then the set of attainable elements $(y^S, y^B, w) \in Y$ is bounded. And for each $j\prime \in F$, the set of $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ is bounded.*

Proof: Appendix

Choose $C \in R, C > 0$ so that $C > |(y^S, y^B, w)|$ (note the strict inequality) for all attainable $(y^S, y^B, w) \in Y$, and so that $C > |(y^{j\prime S}, y^{j\prime B}, w^{j\prime})|$ (note the strict inequality) for all $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ for all $j\prime \in F$. That is, there is a constant C so that all of the attainable points in Y and in any Y^j are strictly contained in a ball of radius C centered at the origin. Let $\Psi \subseteq R^{N(N-1)}$ be a closed ball in $N(N-1)$ space of radius C centered at the origin. Let $\Psi^2 = \Psi \times \Psi$, $\quad \Psi^3 = \Psi^2 \times \Psi$. Let $\bar{\Psi}$ be a closed ball of radius $C(\#H + \#F)$ in $N(N-1)$ space. That is, $\bar{\Psi}$ is a closed ball of radius sufficiently large to encompass all #H-fold plus #F-fold sums taken from Ψ . Let $\bar{\Psi}^2$ be a closed ball of radius $C(\#H + \#F) \in 2N(N-1)$ space.

Lemma 2. *Let $p \in \mathbb{R}_+^{2N(N-1)}$. $B(p)$, $S^j(p)$, and $S^{j\dagger}(p)$ are homogeneous of degree zero in p.*

Proof: Appendix.

As a consequence of Lemma 2, it is sufficient to consider $p \in \Delta$ where Δ is the unit simplex in $\mathbb{R}_+^{2N(N-1)}$. Let $p \in \Delta, p = (q, \pi)$. Firm j's provisionally bounded supply decision then is:

$$\tilde{S}^{j\dagger}(p) \equiv \{(y^S, y^B, w) | (y^S, y^B, w) = \arg\max(q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j)$$
$$\text{subject to } (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}] \cap \Psi^3)\}$$

Firm j's provisionally bounded marketed supply behavior then is:

$$\tilde{S}^{j}(p) \equiv \{(y^S, y^B) | (y^S, y^B, w) = \arg\max(q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j)$$
$$\text{subject to } (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}] \cap \Psi^3)\}$$

Let $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$. (y^{jS}, y^{jB}) is said to be attainable if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ so that (y^{jS}, y^{jB}, w^j) is attainable.

Lemma 3. *Assume P.I through P.IV. Then $\tilde{S}^{j\dagger}(p)$ and $\tilde{S}^j(p)$ are nonempty, convex-valued, and upper hemicontinuous throughout $p \in \Delta$. Let $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$ be attainable. Then $(y^{jS}, y^{jB}) \in S^j(p)$. Let $(y^{jS}, y^{jB}, w^j) \in \tilde{S}^{j\dagger}(p)$ be attainable. Then $(y^{jS}, y^{jB}, w^j) \in S^{j\dagger}(p)$.*

Proof: Appendix.

7 Households

There is a finite set of households H with typical element $i \in H$. Household i's possible consumption set is $W^i \subseteq R^{2N(N-1)}$. We can specify W^i more precisely. Define

$$W^i \equiv \{x' \in R^{2N(N-1)} | \sum_{l=1, k\neq l}^{N} (x'^B(k,l) + x'^S(k,l)) \geq 0, \text{ for each } k = 1, 2, \ldots, N\}.$$

(The notation x' is intended to avoid confusion with household transaction offers.) Household i has an endowment $r^i \in R_+^{2N(N-1)}$. It is sufficient to characterize household preferences by a well-behaved continuous concave utility function u^i on $W^i \subseteq R^{2N(N-1)}$.

The following conditions on household trading and preferences are familiar in the general equilibrium theory, with adaptation to the current setting. They are intended to parallel their counterparts in Starr (2011). (C.I), (C.II), (C.III) are fulfilled by the definition of W^i above.

(C.I) W^i is closed and nonempty.

(C.II) $W^i \subseteq R^{2N(N-1)}$ is bounded below and unbounded above.

(C.III) W^i is convex.

(C.IV) (nonsatiation) Let $x \in W^i$. Then there is $x'' \in W^i$ so that $u^i(x'') > u^i(x)$.

(C.V) (continuity) $u^i : W^i \to R$. u^i is well defined, and continuous.

(C.VI) (convexity of preferences) u^i is quasi-concave. That is, let there be $x, x' \in W^i$ so that $u^i(x') \geq u^i(x), 0 \leq \alpha \leq 1$. Then $u^i(\alpha x + (1-\alpha)x') \geq u^i(x)$.

(C.VII) (strict positivity of income and endowment) $r^i \in W^i$. r^i is strictly positive co-ordinatewise, $r^i >> 0$, where 0 is the zero vector in $R^{2N(N-1)}$.

Household i has a share Θ^{ij} of firm j . Firm j makes a distribution to shareholders $[(y^{jS}, y^{jB} + w^j)] \in R^{2N(N-1)}$ of which i receives $\Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$ leading to a total of dividend distributions $\sum_{j \in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$. i makes trades $x^i \in R^{2N(N-1)}$. $x^i = (x^{iS}, x^{iB})$. $x^{iB} \geq 0, x^{iB} \in R^{N(N-1)}$, is the vector of i's purchases. $x^{iS} \leq 0, x^{iS} \in R^{N(N-1)}$ is the vector of i's sales. The household sells at bid prices, and buys at ask prices. (Informally, it buys retail and sells wholesale.)

The budget constraint on household transactions is $x^i \in B(p)$. Let $\hat{y} = (y^{1S}, y^{1B}, w^1, \ldots, y^{jS}, y^{jB}, w^j, \ldots.) \in Y^1 \times Y^2 \times \ldots \times Y^{\#F}$. The household opportunity set is defined as:

$$A^i(p, \hat{y}) \equiv B(p) + \{r^i\} + \sum_{j \in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$$

Demand behavior is given by:

$$\begin{aligned} D^i(p, \hat{y}) &= \{x \in B(p) | x = \arg\max u^i(x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)) \\ &\qquad \text{subject to } (x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)) \in W^i\} \\ &= \Big\{\{\arg\max u^i(x) \text{ for } x \in [A^i(p, \hat{y}) \cap W^i]\} \\ &\qquad - \{r^i + \sum_{j \in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\}\Big\} \end{aligned}$$

The provisionally bounded household opportunity set is defined as

$$\tilde{A}^i(p, \hat{y}) \equiv \{x \in B(p) | \big[x + r^i + \sum_{j \in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\big] \cap \Psi^2\}$$

Provisionally bounded household demand behavior is described as:

$$\begin{aligned}\tilde{D}^i(p,\hat{y}) &= \{x\in B(p)|x = \arg\max u^i(x+r^i+\sum_j \Theta^{ij}(y^{jS},y^{jB}+w^j))\\ &\qquad \text{subject to } [x+r^i+\sum_j \Theta^{ij}(y^{jS},y^{jB}+w^j)]\in \Psi^2\}\\ &= \Big\{\{\arg\max u^i(x) \text{ for } x\in[\tilde{A}^i(p,\hat{y})\cap W^i]\}\\ &\qquad -\{r^i+\sum_{j\in F}\Theta^{ij}[(y^{jS},y^{jB}+w^j)]\}\Big\}.\end{aligned}$$

Let $\Psi^{3\#F}$ be the #F-fold Cartesian product of Ψ^3 .

Lemma 4. *Let $\hat{y}\in\Psi^{3\#F}$.*
(i) Then $\tilde{D}^i(p,\hat{y})$ is nonempty and homogeneous of degree zero in p. $\tilde{A}^i(p,\hat{y})$ is continuous (upper and lower hemicontinuous) throughout $\Delta\times\Psi^{3\#F}$ and convex-valued. $\tilde{D}^i(p,\hat{y})$ is upper hemicontinuous throughout Δ and convex-valued.
(ii) Let $x^i\in\tilde{D}^i(p,\hat{y})$ be attainable. Then $x^i\in D^i(p,\hat{y})$.

Proof: Appendix.

8 General Equilibrium

A market equilibrium is a vector of prices (q^*,π^*), a vector $c^{i*}\in W^i$, and $x^{i*}\in R^{2N(N-1)}$ for each household, $i\in H$, and a vector $(y^{jS*},y^{jB*},w^{j*})\in Y^j$ for each firm $j\in F$, such that

(i) $c^{i*}=r^i+x^{i*}+\sum_{j\in F}\Theta^{ij}[(y^{*jS},y^{*jB}+w^{*j})]$ is maximal with respect to u^i in W^i subject to $x^i\in B(p^*)$ at $p^*=(q^*,\pi^*)$,

(ii) (y^{jS*},y^{jB*},w^*) maximizes $[(q,q+\pi)\cdot(y^{jS},y^{jB}+w^j)]$ subject to $(y^{jS},y^{jB})\in B(p^*)$, and $(y^{jS},y^{jB},w^j)\in Y^j$.

(iii) $\sum_i(x^{iS*},x^{iB*})+\sum_j(y^{jS*},y^{jB*})\le 0$ co-ordinatewise

(iv) $q^*\ge 0,\pi^*\ge 0$ (the inequalities hold co-ordinatewise).

Note that in (iii) supplies enter with negative signs and demands with positive signs co-ordinatewise.

Let $\hat{x}\in\Psi^{2\#H};\hat{y}\in\Psi^{2\#F}$.
Excess demand is defined as $Z(\hat{x},\hat{y})\equiv\sum_i(x^{iS},x^{iB})+\sum_j(y^{jS},y^{jB})$.

(P.I) through (P.IV) and (C.I) through (C.VII) are sufficient to assure existence of a market clearing equilibrium price vector $p = (q, \pi)$. (C.VII) is used to avoid discontinuities at the boundary.

Lemma 5 (Weak Walras Law). *Let* $p = (q, \pi) \in \Delta$. *Let* $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$ *and let* $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$.
Then $q \cdot [\sum_i x^{iS} + \sum_j y^{jS}] + (q + \pi) \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.
Equivalently, $q \cdot [\sum_i x^{iS} + \sum_j y^{jS} + \sum_i x^{iB} + \sum_j y^{jB}] + \pi \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.

Proof: Appendix.

Theorem 1. *Assume (P.I) through (P.IV) and (C.I) through (C.VII). Then the economy has a competitive equilibrium.*

Proof. Let $\prod_j, \prod_i$ indicate multiple Cartesian product.

Let $\hat{x} \in \prod_{i \in H} \tilde{D}^i(p, \hat{y}) \subseteq \Psi^{2\#H}$; $\hat{x} \equiv (x^{1S}, x^{1B}; x^{2S}, x^{2B}; ...; x^{\#HS}, x^{\#HB})$

$\hat{y} \in \prod_{j \in F} \tilde{S}^{j\dagger}(p) \subseteq \Psi^{3\#F}$; $\hat{y} \equiv (y^{1S}, y^{1B}, w^1; y^{2S}, y^{2B}, w^2; ...; y^{\#FS}, y^{\#FB}, w^{\#F})$

Excess demand is defined as: $Z(\hat{x}, \hat{y}) = \sum_{i \in H} (x^{iS}, x^{iB}) + \sum_{j \in F} (y^{jS}, y^{jB})$; $z = (z^S, z^B)$

Let $\Gamma(z) \equiv \{(q', \pi') \in \Delta | (q', \pi') = \arg\max_{(q,\pi) \in \Delta} (q, \pi) \cdot (z^S + z^B, z^B)\}$ be the price adjustment correspondence.

Let $T(p, \hat{x}, \hat{y}, z) \equiv \Gamma(z) \times \prod_{i \in H} \tilde{D}^i(p, \hat{y}) \times \prod_{j \in F} \tilde{S}^{j\dagger}(p) \times Z(\hat{x}, \hat{y})$.

That is, T is a set-valued mapping
$T : \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2 \rightarrow \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$.

Note that $\Gamma, \tilde{D}^i, \tilde{S}^{j\dagger}$, and $Z(\hat{x}, \hat{y})$ are each well defined, upper hemicontinuous, and convex-valued throughout $\Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$. Then the proof will apply the Kakutani Fixed Point Theorem, to generate a fixed point, $(p^\circ, \hat{x}^\circ, \hat{y}^\circ, z^\circ)$.

Lemma A. *The correspondence* $T(p, \hat{x}, \hat{y}, z)$ *is non-empty, upper hemicontinuous and convex valued.*

Proof of Lemma A: $\tilde{D}^i(p, \hat{y})$ is non-empty, convex valued and upper-hemicontinuous by Lemma 4. Similarly, each $\tilde{S}^j$ is non-empty, convex valued and upper-hemicontinuous by Lemma 3. $Z(\hat{x}, \hat{y})$ is an additive function between finite dimensional spaces, hence continuous. Trivially it is also non-empty and convex valued since it is the finite sum of correspondences that satisfy these properties. Finally Γ is nonempty, since $(q, \pi) \cdot (z^S + z^B, z^B)$ is a continuous function on a compact set Δ, hence, by the Weierstrass theorem, it achieves a maximum. Because $(q, \pi) \cdot (z^S + z^B, z^B)$ is linear in the z's, the set of maximizers of that function is a convex set. Hence Γ is convex valued.

Now we show that Γ is upper-hemicontinuous. Think of z as the parameter affecting the continuous function $f(p,z) := (q,\pi)\cdot(z^S + z^B, z^B)$, the maximand. Notice that the constraint set Δ can be viewed as a constant, compact correspondence of z. Hence, Δ, as a correspondence, is trivially continuous. Then, by the maximum theorem, the set of maximizers with respect to $p = (q,\pi)$ of $f(z,p)$, is an upper-hemicontinous correspondence of the parameter z. Since we defined $\Gamma(z)$ as such a set of maximizers, the desired result follows.

Notably, the finite cartesian product of correspondences preserves upper-hemicontinuity, convex valued-ness and non-emptiness. This completes the proof of Lemma A.

The proof of the theorem continues. Δ is clearly a compact, convex, non-empty set. The above lemma satisfies the assumptions of the Kakutani fixed point theorem, hence T has a fixed point, i.e. $(p^o, x^o, y^o, z^o) \in T(p^o, x^o, y^o, z^o)$.

We now show that (p^o, x^o, y^o, z^o) is a market clearing equilibrium.

By the Weak Walras Law, Lemma 5, $(q^\circ, \pi^\circ)\cdot(z^{\circ S} + z^{\circ B}, z^{\circ B}) \leq 0$.

Notice that $(q^\circ, \pi^\circ) \geq 0$ and (q°, π°) is $\text{argmax}_{(q,\pi)\in\Delta}[q\cdot(z^{oS} + z^{oB}) + \pi\cdot z^{oB}]$ so $z^{^o} \leq 0$. If the inequality were not to hold, the maximand could be increased by increasing the price of the positive component of z. So it must be the case that $z^o \leq 0$, co-ordinatewise.

We have $x^{i\circ} \in \tilde{D}^i(p^\circ, \hat{y}^\circ)$, $y^{j\circ} \in \tilde{S}^j(p^\circ)$. Note the tilde $\tilde{}$ notation. We now seek to demonstrate that, for each $i \in H$ and each $j \in F$, $x^{i\circ} \in D^i(p^\circ, \hat{y}^\circ)$, $y^{j\circ} \in S^j(p^\circ)$. Recall $z^o = \sum_{i\in H} x^{io} + \sum_{j\in F} y^{jo}$ where $x^{io} \in \tilde{D}^i(p^o, \hat{y}^o)$ and $y^{jo} \in \tilde{S}^j(p^o)$.

But $\sum_{i\in H} x^{io} + \sum_{j\in F} y^{jo} \leq 0$, so $x^{io}, i \in H$ is attainable, so $|x^{io}| < C$. But $|x^{io}| < C$ and $x^{io} \in \tilde{D}^i(p^{^o}, \hat{y}^o)$ implies that the constraint to length C is not binding, so by Lemma 4, $x^{io} \in D^i(p^o, \hat{y}^o)$. Similarly, by Lemma 3, $y^{jo} \in S^j(p^o, \hat{x}^o)$. Hence markets clear and the households and firms are optimizing subject to budget and technology constraints. The length constraint is not binding. The price and allocation is a general equilibrium.

□

9 Commodity Money

The distinctive result here is to demonstrate the existence and transaction function of money, endogenously as the result of elementary properties of the economy and its equilibrium. Commodity moneys occur endogenously in the market equilibrium reflecting the constraints of *quid pro quo* embodied in $B(p)$.

In a competitive general equilibrium, let $x^{iS}(k,\ell) < 0$, $x^{iB}(k,m) > 0$, for some i, k, ℓ, m. That is, household i both buys and sells good k in exchange for two different goods. Then k is a medium of exchange, a commodity money. How can we distinguish between commodity k's role as a medium of exchange and arbitrage in k ? There will be no arbitrage in a general equilibrium. Any profitable arbitrage will be infinitely profitable at infinite scale. Hence it cannot occur in equilibrium. In transactions where a household both buys and sells the same commodity, then that commodity is a medium of exchange, commodity money.

Recall that $x^{iS}(k,l) \leq 0$ is i's sales of k for l. $x^{iB}(k,l) \geq 0$ is i's purchases of k for l. Then $\sum_{l \neq k} -x^{iS}(k,l) \ + \sum_{l \neq k} x^{iB}(k,l) \geq 0$ is i's gross trade in k. $\sum_{l \neq k} x^{iS}(k,l) \ + \sum_{l \neq k} x^{iB}(k,l)$ is i's net trade in k. Then $\sum_{l \neq k} -x^{iS}(k,l) \ + \sum_{l \neq k} x^{iB}(k,l) - |[\sum_{l \neq k} x^{iS}(k,l) \ + \sum_{l \neq k} x^{iB}(k,l)]|$ is the volume of transactions in good k commodity money, flow of good k as medium of exchange, gross transactions minus net transactions.

10 Alternative Approaches

10.1 Fiat Money

Government-issued fiat money can occur and maintain positive value through acceptability in payment of taxes, Goldberg (2012), Knapp (1905), Lerner (1947), Smith (1963), Ostroy & Starr (1974), Starr (2012). It can become a common medium of exchange through low transaction cost or thick market externality, (Rey 2001).

10.2 Money Demand and Store of Value

This paper is a single period model, so it does not treat directly the stock of money held over time. Particularly germane in this context is the Baumol (1952)-Tobin (1956) model. Treating that model in an Arrow-Debreu general equilibrium context is beyond the scope of this paper, but the outlines of an interpretation can be developed. Note that the literature includes some notable work on the topic, Hahn (1971), Heller (1974), Heller & Starr (1976), Kurz (1974), Starrett (1973).

The first step would be to time-date all of the commodities, designating each commoditiy by its date of availability. A typical commodity will now be denoted $x(k,l,t)$. Dating a commodity is conventional in the Arrow-Debreu model, Debreu (1959), but here it is simplest to avoid the notion of futures markets, so transactions at date t should be only of goods dated t. It is possible to introduce a storage technology for each household, but it it is probably easier to suppose that all commodities are — if not consumed or sold — durably carried over from one period to the next. Let $c^i(k,l,t)$ be household i's consumption of $x(k,l,t)$ at date t. Then the carryover of $x(k,l,t)$ from t to $t+1$ would be $d^i(k,l,t) \equiv d^i(k,l,t-1) + x^{Bi}(k,l,t) + x^{Si}(k,l,t) - c^i(k,l,t)$. In addition, there should be nonnegativity requirements, $d^i(k,l,t) \geq 0$. The nonnegativity requirement means that positive inventories may be carried forward, but not deficits — one cannot have a debt balance in $x(k,l,t)$.

A finite horizon allows the analysis to proceed. The finite horizon leads to issues at and near the terminal period, where media of exchange may lose their value. Since the present model lends itself to commodity money, that is not so pressing a problem as for fiat money. In the case of fiat money, it may be sufficient to assume that the desirability of paying taxes in the terminal period remains significant.

10.3 Matching Models

How does commodity money in the trading post model compare to commodity money in the trade search literature, (Kiyotaki & Wright 1989, 1993)? In the trading post model, the commodity money can arise in equilibrium because the firm's transaction cost structure can support a bid-ask spread that is different for each commodity pair. There is a similarity to the transaction costs in Kiyotaki & Wright (1989). In that model, trade costs endogenously arise as a function of both agents' trading strategies and storage costs. The storage cost is a deep parameter of the model which determines the type of commodity money equilibria that can arise.

In the case where there is a thick market externality, Rey (2001) and below section 12, in the trading post model, there is another construction here paralleling the matching model. In the trading post model, coordination around a commodity money would be mediated by the effect of trade volumes in the firm's technology and depends on market forces. Kiyotaki & Wright (1989) uses a belief system about other agents' adoption of commodity money as a self-verifying coordination device, designating *speculative* equilibria. Hence, in both models, the nature of commodity money equilibria can be constructed to be self-enforcing, obtaining the same qualitative economic result.

Finally, in both models, commodity money equilibria may support more than one commodity money at a time, albeit for households' with different sets of preferences.

An additional framework is the one presented by Howitt (2005) which combines elements of cash-in-advance and search theoretic models. Our treatment shares the existence of a monetary equilibrium under very general assumptions on preferences. In Howitt (2005) monetary and commodity *quid-pro-quo* transactions may coexist. Conversely in the trading post model, full monetary equilibrium may arise even when there is double coincidence of wants, analogous to Howitt (2005) robust monetary equilibrium, which is not always guaranteed to exist. We believe this latter feature reflects the almost exclusive monetary nature of real world transactions.

An early account of money in a strategic setting is also given in Shubik (1973). The author notes that "money enters into trade in a way that distinguishes it strategically from other commodities". The present paper incorporates this lesson in that commodity money is an equilibrium notion that arises from households' trading strategies as opposed to households' preferences.

11 Conclusion

The Arrow & Debreu (1954) model summarizes each agent's budget as a single equation: the value of sales minus the value of purchases is a firm's profit; the value of endowment plus the value of dividends is the limit of a household's purchases. That is a powerful simplification. The alternative, presented here, is to recognize a separate budget at each of many distinct transactions. That treatment generates a need for a carrier of value between transactions. Hence a model of commodity money.

12 Postscript: Thick Market Externality

Menger (2002) writes:

> **The emergence of media of exchange** In this situation it will occur to everyone bringing goods to market to exchange them for goods meeting his special need that if his goal is not directly attainable because of the limited marketability of his goods, he could exchange them for goods that are considerably more marketable than his own even if he has no direct need for them. To be sure, he does not thus attain the ultimate goal of his intended exchange (acquisition of the goods he especially needs!) immediately and directly. But he comes closer to this goal. In the roundabout way of an intermediary exchange (by giving up his less marketable goods for more marketable ones), he gets the chance of reaching his ultimate goal in a surer and more economical way than by restricting himself to direct exchange ... at first only some economic agents will have recognized the advantage for their economic activities arising from the procedure just described - an advantage actually independent of the general recognition of a good as medium of exchange...
>
> When the relatively most saleable commodities have become "money," the great event has ... the effect of substantially increasing their originally high saleableness.

Rey (2001) formalizes this concept as a 'thick markets externality.' High trading volume generates low transaction costs. Then the resulting market equilibriun in intermediary commodities concentrates on high volume instruments. The medium of exchange becomes unique in equilibrium.

This concept is illustrated in the following example.

Define the volume of trade at ask prices $v(l, l')$ for commodity l and l' at trading post (l, l') as:

$$v(l, l') := |\sum_{h \in H} x^{hB}(l, l')| + |\sum_{h \in H} x^{hB}(l', l)|$$

At a typical trading post, the transaction cost, assessed in the good demanded, is described as

$$w^j(l, l') = \left[\frac{1}{1 + v(l, l')}\right] (|y^{jB}(l, l')|)$$

This cost structure reflects a thick markets externality. High offer volume generates low marginal and average transaction cost. That is, transaction costs are linear in the transactions executed at the trading post but inversely proportional to the total volume of household offers at the trading post (l, l'). Marginal cost pricing of the transaction cost markup gives

$\pi(l, l') = \left[\frac{1}{1+v(l,l')}\right]$

Going back to the example of Section 4, let $\pi(i,j) = 1 = \pi(j,i)$, $\pi(j,k) = \pi(k,j) = \pi(k,i) = \pi(i,k) \approx \frac{1}{100Q} = \epsilon$. This pricing leads to the pattern of trade in the example of Section 4.

Considering the high markup at $\{i,j\}$ households of type j do not trade directly for i. Rather they trade j for k and k for i. k becomes commodity money. Why indirect trade? The markup at $\{k,j\}$ and $\{k,i\}$ is much lower, ϵ. Where do the markups come from? High volume at $\{k,j\}$ and $\{k,i\}$ creates low transaction costs and consequent low (competitive) markup considering the thick markets externality. Thus high volume trade and low volume trade each become self-confirming as discussed in Tobin (1980).

Arrow & Hahn (1971) present sufficient conditions for establishing existence of general equilibrium in the presence of external effects. The result can follow under the usual continuity and convexity conditions, but it requires additional structure focusing on compactness of tastes and technology as external effects vary. We leave this to future research.

13 Appendix: Proofs of Lemmas

13.1 Lemma 1

Lemma 1. *Assume P.I through P.IV. Then the set of attainable elements* $(y^S, y^B, w) \in Y$ *is bounded. And for each* $j\prime \in F$ *, the set of* $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ *attainable in* $Y^{j\prime}$ *is bounded.*

We recall the specification of attainable set. A vector of exchanges (y^S, y^B, w) is said to be attainable if it satisfies:

$$\sum_{j\in F}\sum_{l} y^{jS}(k,l) + \sum_{j\in F}\sum_{l} y^{jB}(k,l) \le \sum_{i\in H}\sum_{l} r^i(k,l)$$
$$-\sum_{j\in F}\sum_{l} w^j(k,l) \le \sum_{i\in H}\sum_{l} r^i(k,l)$$

The first condition says that the net purchase of resources cannot exceed the endowment of the economy as a whole. The second inequality says that total disbursement in transaction costs cannot exceed the total endowment of the economy.

Now a vector of exchanges $y^{j'} := (y^{j'S}, y^{j'B}, w^{j'})$is attainable in $Y^{j'}$ if, for each $j \neq j'$ there is $y^j := (y^{jS}, y^{jB}, w^j) \in Y^j$ such that $y^{j'} + \sum_{j\neq j'} y^j$ is attainable.

Recall P.I, P.II, P.III, P.IV.

Proof: The proof proceeds by contradiction. We will denote $y^{\nu j'} := (y^{S\nu j'}, y^{B\nu j'}, w^{\nu j'})$ First suppose there exist indeed one $j' \in F$ such that its attainable set is unbounded.

Then, there must be a sequence $\{y^{\nu j}\} \subset Y^j$, for each $j \in F$ such that the following properties hold:
i) $|(y^{S\nu j'}, y^{B\nu j'}, w^{\nu j'})| \to +\infty$
ii) $y^{\nu j} \in Y^j \ \forall j \in F$
iii) $y^{\nu} := \sum_{j\in F} y^{\nu j}$ are all attainable.

set $\mu_v := \max_{j\in F} |y^{\nu j}|$. Because for at least one firm j', the set of attainable y is unbounded we immediately have $\mu_v \to +\infty$ as $\nu \to +\infty$.

By P.II, $0 \in Y^j$. By P.I, convexity of the Y^j, for ν sufficiently large, we can write $\tilde{y}^{vj} = \frac{1}{\mu_\nu} y^{\nu j} + (1 - \frac{1}{\mu_\nu})0 \in Y^j$. The attainability condition requires:

$\sum_{j\in F}\sum_l [\tilde{y}^{Svj}(k,l) + \tilde{y}^{Bvj}(k,l)] \leq \frac{1}{\mu_\nu}\sum_{i\in H}\sum_l r^i(k,l)$

$\sum_{j\in F}\sum_l \tilde{w}^{vj}(k,l) \geq -\frac{1}{\mu_\nu}\sum_{i\in H}\sum_l r^i(k,l)$

Observe that by definition the sequences $\tilde{y}^{\nu j}$ are bounded since $|\tilde{y}^{\nu j}| \leq 1$. Consider the vector $(\tilde{y}^{\nu 1}, \tilde{y}^{\nu 2}, \cdots \tilde{y}^{\nu \# F})$. Because this sequence is bounded (immediate to show that it is bounded by $|F|$) there exist a converging sub-sequence that converges to a limit $(\tilde{y}^{o1}, \tilde{y}^{o2}, \cdots, \tilde{y}^{o\#F})$ where $\tilde{y}^{oj} \in Y^j$, by P.III (Y^j closed). The right hand side of the inequalities above imply, in the limit: $-\frac{1}{\mu_\nu}\sum_{i\in H}\sum_l r^i(k,l) \to 0$. We have:

$\sum_{j\in F}\sum_l \tilde{y}^{ojS}(k,l) + \sum_{j\in F}\sum_l \tilde{y}^{ojB}(k,l) \leq 0$

$\sum_{j\in F}\sum_l \tilde{w}^{oj}(k,l) \geq 0$

Hence they imply:

$\sum_{j\in F}\sum_l \tilde{y}^{ojS}(k,l) + \sum_{j\in F}\sum_l \tilde{y}^{ojB}(k,l) + \sum_{j\in F}\sum_l \tilde{w}^{oj}(k,l) \leq 0$ since $\sum_l \tilde{w}^{oj}(k,l)$ is always non-positive. Moreover, from the nonpositivity of $\tilde{w}^{oj}(k,l)$ it must be that

$\sum_l \sum_{j\in F} \tilde{w}^{oj}(k,l) \leq 0$. Together with $\sum_{j\in F}\sum_l \tilde{w}^{oj}(k,l) \geq 0$ it must be the case that:

$\sum_l \sum_{j\in F} \tilde{w}^{oj}(k,l) = 0$

This term, for each k, is the sum of $(N-1)\cdot \#F$ terms. But then, since each $\tilde{w}^{oj}(k,l)$ is non positive it must be the case that for each (l,k), for each $j \in F$, $\tilde{w}^{oj}(k,l) = 0$. Hence, for each $j \in F$ we have: $\tilde{w}^{oj} = 0$. Notice now that, since Y^j is closed for every j, $(\tilde{y}^{ojS}, \tilde{y}^{ojB}, \tilde{w}^{oj}) \in Y^j$. But then, by condition P.IV (i), $\tilde{w}^{oj} = 0$ implies $(\tilde{y}^{ojS}, \tilde{y}^{ojB}, \tilde{w}^{oj}) = 0$.

But then for each $j \in F$, $|\tilde{y}^{vj}| \to |\tilde{y}^{oj}| = 0 \neq 1$ which shows the desired contradiction. That is, there is no such firm j' for which the attainable set is unbounded. Then the set of attainable transactions as a subset of Y^j is bounded for every $j \in F$.

Remark: Incidentally, since the attainable set for every firm j is bounded and there are a finite number F of firms, the attainable set for the whole economy is also bounded.

13.2 Lemma 2

Lemma 2. *Let $p \in \mathbb{R}_+^{2N(N-1)}$. $B(p)$, $S^j(p)$, and $S^{j\dagger}(p)$ are homogeneous of degree zero in p.*

Proof: By inspection.

13.3 Lemma 3

Lemma 3. *Assume P.I through P.IV. Then $\tilde{S}^{j\dagger}(p)$ and $\tilde{S}^{j}(p)$ are nonempty, convex-valued, and upper hemicontinuous throughout $p \in \Delta$. Let $(y^{jS}, y^{jB}) \in \tilde{S}^{j}(p)$ be attainable. Then $(y^{jS}, y^{jB}) \in S^{j}(p)$. Let $(y^{jS}, y^{jB}, w^{j}) \in \tilde{S}^{j\dagger}(p)$ be attainable. Then $(y^{jS}, y^{jB}, w^{j}) \in S^{j\dagger}(p)$.*

Recall P.I, P.II, P.III, P.IV.

$$\begin{aligned}\tilde{S}^{j}(p,\pi) :=& \{(y^{B}, y^{S}) | (y^{B}, y^{S}, w) = \arg\max\ (q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j}) \\ & \text{subject to } (y^{B}, y^{S}, w) \in Y^{j} \cap B(q,\pi) \times R^{N(N-1)} \cap \Psi^{3}\}\end{aligned}$$

Proof. The proof will use the maximum theorem. We first check the hypotheses. Define $C(q,\pi) := \{(y^{B}, y^{S}, w) \in Y^{j} \cap B(q,\pi) \times R^{N(N-1)}\} \cap \Psi^{3}$. Observe that $C(q,\pi)$ is a continuous correspondence in (q,π) because $B(q,\pi)$ is a continuous correspondence. Moreover, for any $(q,\pi), C(q,\pi)$ is closed and bounded. Closed-ness follows from definition as the intersection of closed sets. Bounded-ness follows from Ψ^{3} being bounded. By the Heine-Borel theorem $C(q,\pi)$ is compact. $C(q,\pi)$ is compact valued. Moreover recall that $(q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j})$ is a continuous function of p. Then, by the maximum theorem, the correspondence
$C^{*}(q,\pi) := \{\arg\max\ (q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j}) | (y^{B}, y^{S}, w) \in C(q,\pi)\}$ is nonempty, compact valued and upper hemicontinous. It remains to show that $C^{*}(q,\pi)$ is convex valued. Recall that $(q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j})$ is linear in (y^{jS}, y^{jB}, w^{j}), implying that the upper level set is convex. Fix (q,π). Let c^{*} be the maximum for $(q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j})$ on $C(q,\pi)$. The upper level set $\{(y^{B}, y^{S}, w) | (q, q+\pi)\cdot(y^{jS}, y^{jB}+w^{j}) \geq c,\ (y^{B}, y^{S}, w) \in C(q,\pi)\} = C^{*}(q,\pi)$ is convex. Because the choice of (q,π) was arbitrary we have convex valued-ness of the $C^{*}(\cdot)$ correspondence. Recognize that $C^{*}(q,\pi) = \tilde{S}^{j\dagger}(q,\pi)$ which finishes the proof of the first part.

Let $(y^{\circ S}, y^{\circ B}, w^{\circ}) \in \tilde{S}^{j\dagger}(p)$. Suppose contrary to the lemma there is $(y^{S}, y^{B}, w) \in Y^{j}$, $(y^{S}, y^{B}, w) \in S^{j\dagger}(p)$ so that $(q, q+\pi)\cdot(y^{S}, y^{B}+w) > (q, q+\pi)\cdot(y^{\circ jS}, y^{\circ jB}+w^{\circ j})$. Then for $1 > \alpha > 0, \alpha \neq 0, 1$, sufficiently large, there is $\alpha(y^{\circ S}, y^{\circ B}, w^{\circ}) + (1-\alpha)(y^{S}, y^{B}, w) \in Y^{j} \cap \Psi^{3}$ by convexity of Y^{j}.
But then $(q, q+\pi)\cdot[\alpha(y^{\circ S}, y^{\circ B}+w^{\circ}) + (1-\alpha)(y^{S}, y^{B}+w)] > (q, q+\pi)\cdot(y^{\circ jS}, y^{\circ jB}+w^{\circ j})$; a contradiction.

□

13.4 Lemma 4

Lemma 4. *Let $\hat{y} \in \Psi^{3\#F}$.*
(i) Then $\tilde{D}^{i}(p,\hat{y})$ is nonempty and homogeneous of degree zero in p. $\tilde{A}^{i}(p,\hat{y})$ is continuous (upper and lower hemicontinuous) throughout $\Delta \times \Psi^{3\#F}$ and convex-valued.

$\tilde{D}^i(p,\hat{y})$ *is upper hemicontinuous throughout* Δ *and convex-valued.*
(ii) Let $x^i \in \tilde{D}^i(p,\hat{y})$ *be attainable. Then* $x^i \in D^i(p,\hat{y})$.

Remark: Part (ii) says that if the length constraint in $\tilde{D}^i(p,\hat{y})$ is not a binding constraint then it can be deleted, and x^i is optimizing subject to distribution and budget only, without the requirement that household i limit the size of its plans.

The proof of Lemma 4 takes place in several steps. In order to demonstrate upper and lower hemicontinuity of $\tilde{A}^i(p,\hat{y})$ we'll characterize it as the intersection of many more elementary sets, $\Lambda^i_{l,k}(p,\hat{y})$, each of which is shown to be upper and lower hemicontinuous. We introduce a lemma of Green and Heller (1981) to demonstrate that the intersection — and hence $\tilde{A}^i(p,\hat{y})$ — is continuous. $\tilde{D}^i(p,\hat{y})$ is the set of maximizers of u^i in a translate of $\tilde{A}^i(p,\hat{y})$. Then the theorem of the maximum leads to upper hemicontinuity of $\tilde{D}^i(p,\hat{y})$. A conventional argument focusing on convexity leads to equivalence to $D^i(p,\hat{y})$.

Proof. We recall the definition of the quantities of interest:

$$\tilde{A}^i(p,\hat{y}) := B(p) + \{\textstyle\sum_{j\in F}\Theta^{ij}\left(\hat{y}^{jS},\hat{y}^{jB}+\hat{w}^j\right)+r^i\}\cap\Psi^2$$
$$\tilde{D}^i(p,\hat{y}) = \Big\{\{\arg\max u^i(x)|x\in[\tilde{A}^i(p,\hat{y})\cap W^i]\} - \{r^i+\sum_{j\in F}\Theta^{ij}[(y^{jS},y^{jB}+w^j)]\}\Big\}.$$

where

$$B(p) := \{x\in R^{2N(N-1)}|q(k,l)x^S(k,l)+[q(k,l)+\pi(k,l)]x^B(k,l)$$
$$+q(l,k)x^S(l,k)+[q(l,k)+\pi(l,k)]x^B(l,k)\le 0,\text{for } 1\le k\ne l\le N\}$$

Let $\Psi^{3\#F}$ be the #F-fold Cartesian product of Ψ^3 .

First we will characterize properties of the $\tilde{A}^i(p,\hat{y})$ correspondence. Non-emptiness follows from compactness of Ψ^2, $0\in B(p)$, and $r^i\in\mathbb{R}^{2N(N-1)}_+, r^i >> 0$.
Further, $\sum_j\sum_{l\ne k}\Theta^{ij}\left(\hat{y}^{jS}(k,l),\hat{y}^{jB}(k,l)+\hat{w}^j(k,l)\right)$ is non-negative, by P.IV(ii). Homogeneity of degree 0 follows immediately from homogeneity of degree 0 of $B(p)$ and the definition of $\tilde{A}^i(p,\hat{y})$. Incidentally $\tilde{A}^i(p,\hat{y})$ is compact because it's the (translated) intersection of a compact set and a closed set, so the intersection is closed and bounded and, by Heine-Borel theorem, compact. $B(p)$ is convex by construction. Moreover, $\tilde{A}^i(p,\hat{y})$ is defined as the intersection of convex sets, hence it's convex.

The core of the proof is to show continuity of $\tilde{A}^i(p,\hat{y})$. We will show upper hemicontinuity and lower hemicontinuity separately with the use of an auxiliary lemma. Consider $x\in\tilde{A}^i(p^o,\hat{y}^o)$. Then $x-(\sum_{j\in F}\Theta^{ij}\cdot(\hat{y}^{joS},\hat{y}^{joB}+w^{jo})+r^i)\in B(p^o)$. We introduce a useful construction below.

For any pair (l,k) define the correspondence:

$$\Lambda^i_{l,k}(p,\hat{y}) := \Bigg\{ x \in \mathbb{R}^{2N(N-1)} \text{ such that: }$$
$$x + \left(\sum_{j \in F} \Theta^{ij} \cdot [\hat{y}^{jS}, \hat{y}^{jB} + \hat{w}^j] + r^i \right) \in W^i \text{ and}$$
$$q(k,l) \cdot x^S(k,l) + [q(k,l) + \pi(k,l)] \cdot x^B(k,l)$$
$$+ q(l,k) \cdot x^S(l,k) + [q(l,k) + \pi(l,k)] \cdot x^B(l,k) \leq 0 \Bigg\}$$

Note that $\Lambda^i_{l,k}(p,\hat{y}) = \Lambda^i_{k,l}(p,\hat{y})$.

Remark: For a given $(p,\hat{y})$, $\Lambda^i_{l,k}(p,\hat{y})$ is the space of all trade vectors available to i that respect the budget constraint at trading post (l,k) at the prevailing prices p and economic activity $\hat{y}$. The relevant information for prices and transactions only concerns the (l,k) and (k,l) components (for both bid and ask, wholesale and retail) of the price and transaction vectors because these are the only two commodities traded at the particular trading post. There are $\frac{N(N-1)}{2}$ such trading posts.

We know that, for a trade commodity vector to be feasible it needs to respect all budget constraints at all trading posts simultaneously. As such we can write:

$$\tilde{A}^i(p,\hat{y}) = \bigcap_{l,k} \Lambda^i_{l,k}(p,\hat{y}) \cap \Psi^2$$

Recall that no provisional bound is imposed on each of the $\Lambda_{l,k}$, that is, each of the x in the $\Lambda_{l,k}$ are not required to lie in Ψ^2. Conversely, elements of the $\tilde{A}^i(p,\hat{y})$ are in the provisionally bounded set.

The reason of this construction lies in the following lemma:

Lemma (Green and Heller, 1981). *Let X and Y be subsets of Euclidean space*[1]*. Let $\Gamma_1 : X \rightrightarrows Y$. $\Gamma_2 : X \rightrightarrows Y$. $\Gamma_3 : X \rightrightarrows Y$. $\Gamma_4 : X \rightrightarrows Y$ be correspondences.*

(i) If Γ_1, Γ_2 are two upper hemicontinous closed-valued correspondences such that $\Gamma_1(x^o) \cap \Gamma_2(x^o) \neq \emptyset$ then $\Gamma(x^o) := \Gamma_1(x^o) \cap \Gamma_2(x^o)$ is upper hemicontinuous.

(ii) If Γ_3, Γ_4 are two lower hemicontinous convex-valued correspondences such that int $(\Gamma_3(x^o)) \cap$ int $(\Gamma_4(x^o)) \neq \emptyset$ then $\Gamma(x^o) := \Gamma_3(x^o) \cap \Gamma_4(x^o)$ is lower hemicontinuous.

A plan of the proof of Lemma 4 is :

[1] We note that Y is simply a generic subset of the Euclidean space, it bears no relation with the Y or Y^j in the treatment of the paper.

- 1) Fix the pair (l,k). Show that the correspondence $\Lambda^i_{l,k}(p,\hat{y})$ is upper and lower hemicontinuous in $(p,\hat{y})$.
- 2) Verify that the conditions for Green & Heller (1981) hold.
- 3) Conclude by induction that $\tilde{A}^i(p,\hat{y})$ is upper and lower hemicontinuous.

Fix a pair (l,k). Consider the correspondence $\Lambda^i_{l,k}(p,\hat{y})$ defined above.

1a) **Upper Hemicontinuity of** $\Lambda^i_{l,k}(p,\hat{y})$:
Consider a sequence $(p^\nu,\hat{y}^\nu)\in\Delta\times Y$ such that $(p^\nu,\hat{y}^\nu)\to(p^o,\hat{y}^o)$ and $x^\nu\in\tilde{A}^i(p^\nu,\hat{y}^\nu)$ as well as $x^\nu\to x^o$. We want to show $x^o\in\tilde{A}^i(p^o,\hat{y}^o)$. Suppose not. Because $\tilde{A}^i(p^o,\hat{y}^o)$ is closed we can take an open set U around x^o which guarantees that $U\cap\tilde{A}^i(p^o,\hat{y}^o)=\emptyset$. But then, we must have:

$$\begin{aligned}&q^o(k,l)\cdot x^{oS}(k,l)+[q^o(k,l)+\pi^o(k,l)]\cdot x^{oB}(k,l)\\&+q^o(l,k)\cdot x^{oS}(l,k)+[q^o(l,k)+\pi^o(l,k)]\cdot x^{oB}(l,k)>\epsilon\end{aligned}$$

with $\epsilon>0$, because x^o must be unfeasible at $(p^o,\hat{y}^o)$. But then, since each of the $x^\nu\in\tilde{A}^i(p^\nu,\hat{y}^\nu)$ we have, by the definition of the budget set:

$$\begin{aligned}&q^\nu(k,l)\cdot x^{\nu S}(k,l)+[q^\nu(k,l)+\pi^\nu(k,l)]\cdot x^{\nu B}(k,l)\\&+q^\nu(l,k)\cdot x^{\nu S}(l,k)+[q^\nu(l,k)+\pi^\nu(l,k)]\cdot x^{\nu B}(l,k)\le 0\end{aligned}$$

for all ν. But then, by the order limit theorem, we must have $\lim_{\nu\to\infty}LHS_\nu\le 0$. Thus, a contradiction. We conclude that $x^o\in\Lambda^i_{l,k}(p,\hat{y})$ so $\Lambda^i_{l,k}(p,\hat{y})$ is upper hemicontinuous.

Remark: There is a shorter proof that leaves out many of the details. Consider the graph of the correspondence $\Lambda^i_{l,k}$ given by $G_{k,l}:=\{(p,\hat{y},\Lambda^i_{l,k}(p,\hat{y})\}\in\Delta\times\prod_j Y^j\times W^i$. Then the budget constraint at trading post (k,l) can be expressed as a function $f_{k,l}:\Delta\times\prod_j Y^j\times W^i\to\mathbb{R}$ where the explicit formula for f is given in the equation above. Moreover, $G_{k,l}=f^{-1}_{k,l}((-\infty,0])$ when the budget constraint is respected. But $f_{k,l}$ is clearly a continuous function (it only involves sums and multiplications) and since $(-\infty,0]$ is closed in $\mathbb{R}$, it follows that $G_{k,l}$ is also closed in $\Delta\times\prod_j Y^j\times\Psi^3$ since it's the pre-image of a closed set under a continuous function. By the Closed Graph Theorem $\Lambda^i_{l,k}(p,\hat{y})$ is an upper-continuous correspondence which is the desired result.

1b) **Lower Hemicontinuity of** $\Lambda^i_{l,k}(p,\hat{y})$:
Let $(p^\nu,\hat{y}^\nu)\to(p^\circ,\hat{y}^\circ)$, $x^\circ\in\Lambda^i_{l,k}(p^\circ,\hat{y}^\circ)$. We seek $x^\nu\in\Lambda^i_{l,k}(p^\nu,\hat{y}^\nu)$ so that $x^\nu\to x^\circ$. Recall that $x^{\circ S}(k,l)\le 0$, $x^{\circ B}(k,l)\ge 0$.

Here's how we'll construct x^ν . For $k', l' \neq k, l$ the choice of $x^\nu(k', l'), x^\nu(l', k')$ is unrestricted by the specification of $\Lambda^i_{l,k}(p^\nu, \hat{y}^\nu)$ so we can set $x^\nu(k', l'), x^\nu(l', k') = x^\circ(k', l'), x^\circ(l', k')$.

For k, l here's the logic of the construction. Recall that the superscript S denotes i's delivery at bid prices to the trading post. For $x^{\nu S}(k, l)$, how much (k, l) can household i arrange to deliver to the trading post? It is limited by its availability of k from all sources — endowment and dividends. If that's less than $x^{\circ S}(k, l)$ then i should deliver what it can. For ν sufficiently large, the availability will approach $x^{\circ S}(k, l)$. If the availability is greater, then it is sufficient to deliver $x^{\circ S}(k, l)$. Similarly for $x^{\nu S}(l, k)$.

The superscript B denotes i's acquisitions at ask prices from the trading post. For $x^{\nu B}(k, l)$ how much (k, l) can i afford to acquire at $(p^\nu, \hat{y}^\nu)$? That depends mainly on the values just established, $x^{\nu S}(k, l)$, $x^{\nu S}(l, k)$, that i delivers to (k, l), evaluated at p^ν. How does that compare to the value of $x^{\circ B}(k, l)$ and $x^{\circ B}(l, k)$ at p^ν? That's the fraction in the specification for $x^{\nu B}(k, l)$ below. When the budget in the numerator is bigger than the budget in the denominator then i can afford $x^{\circ B}(k, l)$ at p^ν. And for ν sufficiently large, i will converge on fully affordable. When i has an even more ample budget, it is sufficient just to plan on $x^{\nu B}(k, l) = x^{\circ B}(k, l)$. The calculation becomes trickier when budgets are zero, as may occur if prices are zero's. Then $x^{\circ B}(k, l)$ is fully affordable and is set equal to $x^{\nu B}(k, l)$. $x^{\nu S}(k, l), x^{\nu S}(l, k), x^{\nu B}(k, l), x^{\nu B}(l, k)$ are described below.

There are two settings to keep in mind to determine $x^{\nu B}(k, l)$ and $x^{\nu B}(l, k)$, depending on the value at p^ν of $x^{\circ B}(k, l)$ and $x^{\circ B}(l, k)$.

If $[q^\circ(k, l) + \pi^\circ(k, l)] \cdot x^{\circ B}(k, l) + [q^\circ(l, k) + \pi^\circ(l, k)] \cdot x^{\circ B}(l, k) > 0$, then for ν large, $[q^\nu(k, l) + \pi^\nu(k, l)] \cdot x^{\nu B}(k, l) + [q^\nu(l, k) + \pi^\nu(l, k)] \cdot x^{\nu B}(l, k) > 0$ and the fraction in the specification of $x^{\nu B}(k, l)$ and $x^{\nu B}(l, k)$ below is well defined.

On the contrary, when $[q^\circ(k, l) + \pi^\circ(k, l)] \cdot x^{\circ B}(k, l) + [q^\circ(l, k) + \pi^\circ(l, k)] \cdot x^{\circ B}(l, k) = 0$, then the fraction in the description of $x^{\nu B}(k, l)$ below may not be well defined. If so the alternative there applies.

Proposed values of $x^{\nu S}(k, l), x^{\nu S}(l, k)$ follow, below. Recall that typically $x^{\circ S}(k, l) \leq 0$, so the "max" notation below means choosing the smaller absolute value. Under (C.VII) the summations in square brackets (that is [ ]) will typically be positive, and negative when denoted with a minus sign (−).

Let $x^{\nu S}(k, l)$ be defined as the maximum between $x^{\circ S}(k, l)$ and
$-\sum_{m=1, m\neq k}^{N} \left[r^{iS}(k, m) + r^{iB}(k, m) + \sum_{j\in F} \Theta^{ij}[\hat{y}^{\nu jS}(k, m) + \hat{y}^{\nu jB}(k, m) + \hat{w}^{\nu j}(k, m)] \right]$
Simlarly let $x^{\nu S}(l, k)$ be defined as the maximum between $x^{\circ S}(l, k)$ and
$-\sum_{m=1, m\neq l}^{N} \left[r^{iS}(l, m) + r^{iB}(l, m) + \sum_{j\in F} \Theta^{ij}[\hat{y}^{\nu jS}(l, m) + \hat{y}^{\nu jB}(l, m) + \hat{w}^{\nu j}(l, m)] \right]$

Let $x^{\nu B}(k, l)$ be defined as the minimum between $x^{\circ B}(k, l)$ and
$\frac{|q^\nu(k,l)x^{\nu S}(k,l) + q^\nu(l,k)x^{\nu S}(l,k)|}{[q^\nu(k,l)+\pi^\nu(k,l)]x^{\circ B}(k,l) + [q^\nu(l,k)+\pi^\nu(l,k)]x^{\circ B}(l,k)} x^{\circ B}(k, l)$

when the fraction is well defined.

Similarly, let $x^{\nu B}(l,k)$ be defined as the minimum between $x^{\circ B}(l,k)$ and $\frac{|q^\nu(k,l)x^{\nu S}(k,l)+q^\nu(l,k)x^{\nu S}(l,k)|}{[q^\nu(k,l)+\pi^\nu(k,l)]x^{\circ B}(k,l)+[q^\nu(l,k)+\pi^\nu(l,k)]x^{\circ B}(l,k)}x^{\circ B}(l,k)$
when the fraction is well defined.

Then $x^{\nu B}(k,l)$ is the amount of (k,l) to be purchased. It is the smaller of $x^{\circ B}(k,l)$ and the affordable fraction of $x^{\circ B}(k,l)$ based on sales of $x^{\nu S}(k,l)$ and $x^{\nu S}(l,k)$ (each determined in the specification above). The affordable proportion is described in the fraction. The numerator is the budget at bid prices based on the sales. The denominator is the expenditure, evaluated at p^ν of the purchases on the (k,l) market $x^{\circ B}(k,l)$ and $x^{\circ B}(l,k)$. Not all of these terms need be nonzero, but that is endogenous. For $(k',l') \neq (k,l)$, let $x^{\nu S}(k',l') = x^{\circ S}(k',l'), x^{\nu B}(k',l') = x^{\circ B}(k',l')$;
$x^{\nu S}(l',k') = x^{\circ S}(l',k'), x^{\nu B}(l',k') = x^{\circ B}(l',k')$. Then $x^\nu \in \Lambda^i_{k,l}(p^\nu,\hat{y}^\nu)$ and $x^\nu \to x^\circ$. Hence x^ν is the required sequence.

2) **Verifying Conditions for Green and Heller (1981) Lemma:** We now verify the conditions for the Green and Heller (1981) lemma. First, it follows immediately from the definition by a linear inequality that $\Lambda^i_{l,k}(p,\hat{y})$ is closed (it's the inverse image of a closed set $(-\infty,0]$ under a continuous function, hence closed). Moreover, since Ψ^2 is bounded, so is $\Lambda^i_{l,k}(p,\hat{y})$. By the Heine-Borel theorem, $\Lambda^i_{l,k}(p,\hat{y})$ is compact. Now, clearly, $0 \in \Lambda^i_{l,k}(p,\hat{y})$ since the zero transaction vector, 0 , satisfies all the inequalities for any $(p,\hat{y})$. Hence $\Lambda^i_{l,k}(p,\hat{y})$ is not empty. Moreover, $0 \in \Lambda^i_{l,k}(p,\hat{y})$ regardless of the choice of the pair (l,k). Convexity is also immediate for $\Lambda^i_{l,k}(p,\hat{y})$ is the intersection of a set defined by a linear inequality, (which is convex) and the convex set Ψ^2. Now, recall $r^i >> 0$. We need to show that for any given price and dividend distribution, the interior of the correspondence $\Lambda_{l,k}(\hat{p},\hat{y})$ is not empty. To show this, it is sufficient to show that there is a collection $c_{l,k}$ of 5 transaction vector points that are in $\Lambda_{l,k}(\hat{p},\hat{y})$ and are in general position. Recall that the construction of $\Lambda_{l,k}(\cdot)$ does not constrain any of the non-(k,l) coordinates of x, other than requiring them to be in W^i. Hence we can always augment a set of 5 points (as we obtain below) to a set of $2N(N-1)+1$ points $\{c_{l,k}, d\}$ by choosing a collection $\{d\}$ to be (the negative of) the standard basis vectors $-\{e_{l'}\}$ for all the non-(k,l) wholesale coordinates and the standard basis vector $\{e_{l'}\}$ for all the non-(k,l) retail coordinates of $\mathbb{R}^{2N(N-1)}$. Such vectors clearly live in W^i. We show below how to pick the 5 points in the $c_{k,l}$ collection. We emphasize the coordinates for the k,l, all other coordinates, indicated by the dots, are taken to be 0. Without loss of generality we may assume that none of the prices are 0, if they are then take local transaction vectors $x_1 = (\cdots,-100,0,0,0,\cdots), x_2 = (\cdots,0,100,0,0,\cdots), x_3 = (\cdots,0,0,-100,0,\cdots), x_4 = (\cdots,0,0,0,100,\cdots)$ respectively, whenever x_i's price is 0. In general, for nonzero prices take the 0 transaction vector, together with:

$$x_1 = \begin{pmatrix} \vdots \\ -\frac{r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix};$$

$$x_2 = \begin{pmatrix} \vdots \\ -\frac{r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ \frac{[r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')]\cdot q(k,l)}{2\cdot[q(k,l)+\pi(k,l)]} \\ 0 \\ 0 \\ \vdots \end{pmatrix};$$

$$x_3 = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ -\frac{r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ 0 \\ \vdots \end{pmatrix};$$

$$x_4 = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ -\frac{r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ \frac{[r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')]\cdot q(l,k)}{2\cdot[q(l,k)+\pi(l,k)]} \\ \vdots \end{pmatrix}$$

where the $x_{k,l}$ coordinates are listed above and all other coordinates are 0. which are guaranteed to be in general position since $r(k,l), r(l,k) > 0$ and the quantity $\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l') \geq 0$. Moreover, they are clearly in W^i, hence they satisfy the desired conditions. We take $c_{k,l} = \{x_0, x_1, x_2, x_3, x_4\}$. The collection of points $\{c_{l,k}, d\}$ preserves general position. Because $\Lambda^i_{l,k}(p,\hat{y})$ is convex, their convex hull, $\text{co}(\{c_{l,k}, d\})$ will be contained in $\Lambda_{l,k}(\hat{p},\hat{y})$. By construction $\text{co}(\{c_{l,k}, d\})$ is a $2N(N+1)$ dimensional polytope and has non-empty interior, showing the desired requirement.

Hence, there exists an open neighborhood $\mathcal{V}_{k,l}$ (in the subset topology) containing 0. Similarly, for a general correspondence $\Lambda^i_{l',k'}(p,\hat{y})$ there exists a neighborhood $\mathcal{V}_{k',l'}$ containing 0. But then, the neighborhood of 0 given by $\mathcal{V}_{k,l} \cap \mathcal{V}_{k',l'} \neq \emptyset$ and by the definition of neighborhood there exist a non-empty open set $\mathcal{V} \subset \mathrm{int}(\mathcal{V}_{k,l}) \cap \mathrm{int}(\mathcal{V}_{k',l'}) \subset \mathrm{int}(\Lambda^i_{l,k}) \cap \mathrm{int}(\Lambda^i_{l',k'})$ which satisfies the condition in the lemma above. Note that Ψ^2 is a constant correspondence which is closed and trivially upper and lower hemicontinous.

3) Now, by the Green & Heller (1981) lemma and an induction argument, we show that $\tilde{A}^i(p, \hat{y}) = \bigcap_{l,k} \Lambda^i_{l,k}(p, \hat{y}) \cap \Psi^2$ is the finite intersection of closed valued, convex-valued upper and lower hemicontinuous correspondences, hence it's upper and lower hemicontinuous.

We have shown that $\tilde{A}^i(p, \hat{y})$ is a continuous correspondence. Then by continuity of $u^i(\cdot)$ and the theorem of the maximum we have: $C(p, \hat{y}) = \{\arg\max u^i(x) | x \in \tilde{A}^i(p, \hat{y})\}$ is continuous. In this case $C(p, \hat{y})$ is exactly the provisionally bounded demand correspondence $\tilde{D}^i(p, \hat{y})$. This finishes the proof of part (i).

Now a short proof of part (ii) of the lemma. Because $x \in \tilde{D}^i(p, \hat{y})$ and x is attainable, we must have $|x| < C$. We claim $x \in D^i(p, \hat{y})$. Suppose not; then there is $x' \in W^i \cap A(p, \hat{y})$ such that $u^i(x') > u^i(x)$. Then $u^i(\alpha x + (1-\alpha)x') > u^i(x)$ for any $\alpha \in (0, 1)$. But choosing $0 < \alpha < 1$ large enough must imply that $x'' := \alpha x + (1-\alpha)x' \in \tilde{A}^i(p, \hat{y})$. But then, $u^i(x'') > u^i(x)$ and both x and $x'' \in \Psi$. So $x \notin \tilde{D}(p, \hat{y})$, a contradiction. □

13.5 Lemma 5: Weak Walras Law

Lemma 5 (Weak Walras Law). *Let $p = (q, \pi) \in \Delta$. Let $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$and let $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$.*
Then $q \cdot [\sum_i x^{iS} + \sum_j y^{jS}] + (q + \pi) \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.
Equivalently, $q \cdot [\sum_i x^{iS} + \sum_j y^{jS} + \sum_i x^{iB} + \sum_j y^{jB}] + \pi \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.

Proof. Because $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, y)$, it follows from the definition of the demand correspondence that, for each individual household we must have:

$$q \cdot x^{iS} + (\pi + q) \cdot x^{iB} \leq 0$$

Similarly, for each firm j we have $(y^{jS}, y^{jB}) \in \tilde{S}(p)$ so it must be the case that, for each j:

$$q \cdot y^{jS} + (\pi + q) \cdot y^{jB} \leq 0$$

Consider the value of aggregate excess demand:

$$q \cdot \left(\sum_i x^{iS} + \sum_j y^{jS} \right) + (\pi + q) \cdot \left(\sum_i x^{iB} + \sum_j y^{jB} \right)$$
$$= \left(q \cdot \sum_i x^{iS} + (\pi + q) \cdot \sum_i x^{iB} \right) + \left(q \cdot \sum_j y^{jS} + (\pi + q) \cdot \sum_j y^{jB} \right)$$

Since, each of $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$ we have

$$q \cdot \sum_i x^{iS} + (\pi + q) \cdot \sum_i x^{iB} = \sum_i \left(q \cdot x^{iS} + (\pi + q) \cdot x^{iB} \right) \leq 0$$

since if the first inequality holds for each household i, all the more it holds for the sum. Hence, the first term of the right-hand-side is not greater than 0. Similarly, since the second inequality holds for every j, all the more:

$$q \cdot \sum_j y^{jS} + (\pi + q) \cdot \sum_j y^{jB} = \sum_j \left(q \cdot y^{jS} + (\pi + q) \cdot y^{jB} \right) \leq 0$$

so the second term of right-hand-side is also no greater than 0. Finally, combining these two one gets:

$$q \cdot \left(\sum_i x^{iS} + \sum_j y^{jS} \right) + (\pi + q) \cdot \left(\sum_i x^{iB} + \sum_j y^{jB} \right) \leq 0$$

which is the desired result.

□

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